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Topological field theory interpretations and LG representation of $c = 1$ string theory

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Abstract

We analyze the topological nature of $c = 1$ string theory at the self–dual radius. We find that it admits two distinct topological field theory structures characterized by two different puncture operators. We show it first in the unperturbed theory in which the only parameter is the cosmological constant, then in the presence of any infinitesimal tachyonic perturbation. We also discuss in detail a Landau–Ginzburg representation of one of the two topological field theory structures.

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1 Introduction

The topological nature of $c = 1$ string theory, [1], at the self–dual radius has been recently studied by several authors. The main tool in this regard has been provided by the remark that the structure underlying this theory is the 2–dimensional dispersionless Toda lattice hierarchy, whose two series of flow parameters play the role of coupling constants of the purely tachyonic states T_n and T_{-n}. In refs.[2], it has been proposed that T_1 play the role of *puncture operator* and the tachyons be primary fields.

In [3],[4] we have shown that the two–matrix model provides a good description of $c = 1$ string theory. In fact: 1) two–matrix model has allowed us to reproduce all the known results of $c = 1$ string theory (which is not surprising since it is equivalent to the Toda lattice hierarchy constrained with suitable boundary conditions); 2) it has allowed us to introduce additional discrete states, which can be interpreted as the discrete states of $c = 1$ string theory, and to compute their correlation functions; 3) two–matrix model is defined for every genus.

The present paper is specifically devoted to the discussion of the topological nature of $c = 1$ string theory. In [4] we suggested that there may exist another topological field theory (TFT) interpretation in which the puncture operator is T_0, the primary fields are all the pure tachyonic states, while the descendants are the discrete states. Here we show that this interpretation as well as the previous one, [2], with some adjustments, are both legitimate. We prove this in the pure cosmological sector of the theory (in which the only parameter is the cosmological constant), but we substantiate our assertion by showing that the axioms of topological field theory hold in the presence of any (infinitesimal) perturbation by the tachyonic states and by providing a Landau–Ginzburg description of the structure which has T_1 as puncture operator. Actually there is a third topological field theory structure in which T_{-1} is the puncture operator, but this structure is exactly specular to the latter.

With the evidence we provide in this paper we think it is justified to say that $c = 1$ string theory at the self–dual radius (and two–matrix model with it) is a huge topological field theory which is defined at all genera. In the final section we suggests that other smaller TFT's are imbedded in such huge theory, and this perhaps provides a clue to understanding why T_0 and T_1 (or T_{-1}) can be both interpreted as puncture operators.

We remark that so far only TFT's with a finite number of primaries have been fully analyzed and coupled to topological gravity, [6], – we have in mind typically the ADE models, let us generically refer to them as $c < 1$ TFT's. The TFT's we are considering here are of a quite different type: first of all they have an infinite number of primaries, secondly the states are classified according to sl_2 representations (instead of a $U(1)$ charge label). It is therefore non–trivial not only that they satisfy the TFT axioms, but that the coupling to topological gravity occurs with essentially the same rules as in the $c < 1$ models. It is remarkable that all these properties, as well as their generalization at every genus, are embodied in the Toda hierarchy subject to the coupling conditions of the two–matrix models.

The paper is organized as follows. Below we summarize some notations and formulas which will be used throughout the article. In section 2 we introduce the two (three) topological field theory structures announced above in the purely cosmological sector and show that all the axioms of TFT's are satisfied. In section 3 we show the same thing in the presence of any infinitesimal tachyonic perturbation. In section 4 we summarize the equations characterizing the dispersionless Toda hierarchy and restrict it to the $c = 1$ string theory. With this material, in section 5, we present a Landau-Ginzburg (LG) interpretation of the TFT structure in which

T_1 is the puncture operator.

1.1 Genus 0 correlators of the discrete states

We collect in this section some results taken from [3],[4] we will need in the following. We start with some notations. We will label the states of the theory with latin letters: the first latin letters $a, b, c, ...$ will be used to denote integers, while $i, j, k, l, m, n, ...$ will denote positive integers and r, s non–negative integers. The tachyonic states will be denoted T_n, T_{-n}. The cosmological operator T_0 will also be denoted Q. So, in particular, the set $\{T_a\}$ is the same as the set $\{T_n, T_0, T_{-n}\}$. The generic discrete states are denoted $\chi_{r,s}$, and we have the correspondence $\chi_{n,0} = T_n$, $\chi_{0,n} = T_{-n}$ and $\chi_{0,0} = T_0$; the remaining discrete states, with both r and s non–vanishing, are called extra.

Each discrete state is coupled to the theory via a coupling $g_{r,s}$. We use the convention $g_{n,0} \equiv t_{1,n}$ and $g_{0,n} \equiv t_{2,n}$, while $g_{0,0}$ is identified with the cosmological constant x. The model or sector of the whole theory that results when we switch on the couplings $t_{1,1}, ..., t_{1,p}, t_{2,1}, ..., t_{2,q}$ (beside $g_{0,0}$ and $g_{1,1}$) will be called $\mathcal{M}_{p,q}$. Throughout the paper we set $g_{1,1} = -1$. Let us denote by $\mathcal{S}_0$ the model $\mathcal{M}_{0,0}$ restricted by the condition $g_{1,1} = -1$. $\mathcal{S}_0$ is what we mean by purely cosmological sector of the $c = 1$ string theory.

The genus 0 correlation functions (CF's), denoted with $< \cdot >$ throughout the paper, for $\mathcal{S}_0$ are given by

$$< \chi_{n,n} >= \frac{x^{n+1}}{n+1} \tag{1.1}$$

$$< \chi_{r_1,s_1}\chi_{r_2,s_2} >= x^{\Sigma}\frac{M(r_1,s_1)M(r_2,s_2)}{\Sigma} \tag{1.2}$$

where $\Sigma = r_1 + r_2 = s_1 + s_2$ and $M(r, s) = max(r, s)$. This formula also holds when the two labels of χ coincide.

$$< \chi_{r_1,s_1}\chi_{r_2,s_2}\chi_{r_3,s_3} >= x^{\Sigma-1}M(r_1,s_1)M(r_2,s_2)M(r_3,s_3) \tag{1.3}$$

where $\Sigma = r_1 + r_2 + r_3 = s_1 + s_2 + s_3$.

For the n–point functions with n > 3, as is by now well–known, there is more than one possibility. We give here only

$$\begin{aligned} &< \chi_{r_1,s_1}\dots\chi_{r_n,s_n} >= x^{\Sigma-n+2}M(r_1,s_1)\dots M(r_n,s_n)(\Sigma-1)\dots(\Sigma-n+3) \\ &\Sigma = r_1+\dots+r_n = s_1+\dots+s_n \end{aligned} \tag{1.4}$$

if $\Sigma > n - 2$, and vanishes otherwise. This formula holds when there is one label $r_k > s_k$ and $n - 1$ labels $r_k < s_k$.

We remark that the above formulas have been derived for states $\chi_{r,s}$ with r and s not simultaneously vanishing. To obtain CF's involving p insertions of $Q \equiv \chi_{0,0}$, one has simply to differentiate p times with respect to x the corresponding CF without Q insertions. For CF's containing only Q insertions, we have

$$F_h = \chi_h^{(0)}x^{2-2h}, \qquad < Q^n >_h= n!\chi_h^{(n)}x^{2-2h-n} \tag{1.5}$$

where

$$\chi_h^{(n)} = \frac{(-1)^n(2h-3+n)!(2h-1)}{n!(2h)!} B_{2h}$$

Here the label h denotes the genus h contribution. $\chi_h^{(n)}$ is the virtual Euler characteristic of the moduli space of Riemann surfaces of genus h, [5]. We see therefore that Q, the operator coupled to x, is to be interpreted in this context as the puncture operator.

2 TFT interpretations

2.1 First interpretation: puncture operator T_1

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First we recall that a (matter) topological field theory is defined by a (usually finite) set of primary fields ϕ_α, $\alpha = 1, 2, \ldots$. Among them one, say ϕ_1, plays a special role. The n–th point correlators are pure numbers and, in particular, the 3–point functions (in genus 0) $C_{\alpha,\beta,\gamma} =< \phi_\alpha \phi_\beta \phi_\gamma >$ are crucial in the definition of TFT. The metric $\eta_{\alpha,\beta}$ coincides by definition with $C_{1,\alpha,\beta}$ and is required to be invertible. The inverse metric is denoted by $\eta^{\alpha,\beta}$. The last defining property of TFT is the associativity condition

$$\sum_{\lambda,\mu} C_{\alpha,\beta,\lambda}\eta^{\lambda,\mu}C_{\mu,\gamma,\delta} = \sum_{\lambda,\mu} C_{\alpha,\gamma,\lambda}\eta^{\lambda,\mu}C_{\mu,\beta,\delta} \tag{2.6}$$

The coupling of such a theory to topological gravity gives rise to the descendants and is regulated by two types of equations, the puncture equations and the recursion relations. ϕ_1 plays the role of puncture operator.

In the first interpretation of $\mathcal{S}_0$ as a topological field theory T_1 is the puncture operator and $T_n \equiv \chi_{n,0}$ and $T_{-n} \equiv \chi_{0,n}$ are the primary fields, while all the extra states are gravitational descendants. To justify this assertion we have to find the metric and the structure constants of the TFT and prove that they satisfy all the axioms. Moreover we have to define suitable puncture equations and recursion relations

The metric and the structure constants are given by

$$\eta_{a.b} =< T_1 T_a T_b >, \qquad C_{a,b,c} =< T_a T_b T_c > \tag{2.7}$$

where a and b are integers. The only nonzero elements are

$$\eta_{n,-n-1} = \eta_{-n-1,n} =< T_1 T_n T_{-n-1} >= n(n+1)x^n, \quad \eta_{0,1} = \eta_{0,-1} = 1$$

This metric is non–degenerate, the inverse is $\eta^{a,b}$ with

$$\eta^{n,-n-1} = \eta^{-n-1,n} = \frac{x^{-n}}{n(n+1)}, \qquad \eta^{1,0} = \eta^{0,-1} = 1$$

while all the other elements vanish. The associativity condition (2.6) for the structure constants $C_{a,b,c}$ is easily seen to be satisfied once we notice that the only nonvanishing three–point

functions among primaries are

$$\begin{aligned}
&C_{n,m,-n-m} = C_{-n,-m,n+m} =< T_{-n}T_{-m}T_{n+m} >= nm(n+m)x^{n+m-1}\\
&C_{n,-m,m-n} = \begin{cases} nm(n-m)x^{n-1}, & n > m \\ nm(m-n)x^{m-1}, & n < m \end{cases}\\
&C_{0,n,m} \equiv n^2 x^{n-1}\delta_{n+m,0}, \qquad C_{0,0,0} = x^{-1}
\end{aligned} \tag{2.8}$$

The primary fields form the commutative associative algebra $\mathcal{A}_1$

$$T_a T_b = \sum_c C_{a,b}{}^c T_c \tag{2.9}$$

where

$$C_{a,b}{}^c \equiv \sum_d C_{a,b,d}\eta^{d,c}, \qquad\qquad T_0 \equiv Q$$

and we have identified T_1 with the identity of $\mathcal{A}_1$[‡]. To prove $\mathcal{A}_1$ one has to use

$$\begin{aligned}
&C_{n,m}{}^{n+m-1} = \frac{nm}{n+m-1}, \qquad C_{0,n}{}^{n-1} = \frac{n}{n-1}, \quad C_{n,-n}{}^{-1} = n^2 x^{n-1}, \qquad C_{0,0}{}^{-1} = x^{-1}\\
&C_{n,-m}{}^{n-m-1} = \begin{cases} \frac{nm}{n-m-1}x^m, & n > m+1 \\ \frac{nm}{m-n+1}x^{n-1}, & n < m \end{cases}\\
&C_{n,-n+1}{}^0 = n(n-1)x^{n-1}
\end{aligned} \tag{2.10}$$

where $n, m \neq 0$, $C_{a,b}{}^c = C_{b,a}{}^c$, and the other structure constants vanish.

Therefore the set of fields $\{T_{-n}, Q, T_n\}$ define a TFT with puncture operator T_1.

Next let us switch on the coupling to topological gravity, and see whether this TFT fits into the scheme of a TFT coupled to topological gravity [6]. We will see that this is indeed the case by exhibiting the appropriate puncture equations and recursion relations. For the *puncture equations* we start from the string equation, [4],

$$\mathcal{L}^{[1]}_{-1}(2)Z(t;g,x) = -T^{[1]}_{-1}(2)Z(t;g,x)$$

where

$$T^{[1]}_{-1}(2) = \sum_{\substack{i\geq 1\\ j\geq 1}} j g_{i,j}\frac{\partial}{\partial g_{i,j-1}}$$

and differentiate it with respect to $g_{i_1,j_1}, \ldots, g_{i_n,j_n}$. Finally we evaluate it in $\mathcal{S}_0$. We find

$$< T_1 \chi_{i_1,j_1} \ldots \chi_{i_n,j_n} >= \sum_{l=1}^{n} j_l < \chi_{i_1,j_1} \ldots \chi_{i_l,j_l-1} \ldots \chi_{i_n,j_n} > \tag{2.11}$$

This relation is exact, i.e. valid for all genera. This holds provided in the LHS there does not appear the operator T_{-1}, or, which is the same, in the RHS there does not appear Q. This

[‡]In order to verify (2.9) in the correlators one must remember to restore T_1 in the RHS of (2.9)

means the following: *the fields $\chi_{n,m}$, $n,m>0$, are the descendants of $\chi_{n,0}=T_n$; Q has no descendants; T_{-n} is both a primary and a descendant of T_{-1}.*

Let us pass now to the *recursion relations.* They are (in genus 0) and in $\mathcal{S}_0$

$$< \chi_{r,s}\chi_{r_1,s_1}\chi_{r_2,s_2} >= M(r,s)\sum_{l,k} < \chi_{r,s-1}T_a > \eta^{a,b} < T_b\chi_{r_1,s_1}\chi_{r_2,s_2} > \tag{2.12}$$

where the labels k and l are understood to be integers. The proof is very simple. Suppose for example that $r \geq s+1$. Then

$$\mathrm{LHS} = rM(r_1,s_1)M(r_2,s_2)x^{r+r_1+r_2-1}$$

when $r+r_1+r_2 = s+s_1+s_2$ and vanishes otherwise. On the other hand

$$\mathrm{RHS} = r < \chi_{r,s-1}T_{s-r+1} > \eta^{s-r+1,r-s} < T_{r-s}\chi_{r_1,s_1}\chi_{r_2,s_2} >= rM(r_1,s_1)M(r_2,s_2)x^{r+r_1+r_2-1}$$

when $r+r_1+r_2 = s+s_1+s_2$, and vanishes otherwise. The same can be proven for $r \leq s+1$ [§].

We can conclude that we have indeed to do with an unperturbed topological field theory coupled to topological gravity. We call it $\mathcal{T}_1$.

The correlators of $c=1$ string theory were obtained in [3], [4], starting from the W constraints of the two-matrix model (or, equivalently, from the Toda flow equations plus the coupling conditions). We can therefore say that all we have seen in this section (including the recursion relations and the puncture equations) is nothing but a manifestation of the W constraints. There is also a direct way to prove this assertion, as pointed out in ref.[7]. We have already noticed that the puncture equations are valid at all genera. As for the other objects and equations studied in this section, *the natural generalization to higher genus is provided by the W constraints, which are all–genus relations.*

Remark. The $< Q^n >$ correlators are exceptional. They are not determined by the above recursion relation and puncture equations, but by the Toda equation (5.2) of [4] which are the appropriate recursion relation for this kind of correlators.

Although the Landau–Ginzburg formalism will be discussud at length later on, let us complete the presentation of $\mathcal{T}_1$ by anticipating how the algebra of the primary fields is reproduced in the LG formalism.

The primary fields are represented by polynomials in the variable ζ and ζ^{-1}

$$\phi_n \equiv \phi_{n,0} = n\zeta^{n-1}, \qquad \phi_0 \equiv \phi_{0,0} = \zeta^{-1}, \qquad \phi_{-n} \equiv \phi_{0,n} = nx^n\zeta^{-n-1}, \tag{2.13}$$

These objects form a commutative and associative algebra $\mathcal{R}_1$ by simple multiplication. We define the following map

$$\phi_n \leftrightarrow T_n, \qquad \phi_0 \leftrightarrow Q, \qquad \phi_{-n} \leftrightarrow T_{-n}$$

and claim that it is an isomorphism between the algebra $\mathcal{R}_1$ and the field algebra $\mathcal{A}_1$. It is elementary to prove the isomorphism by checking the few non–trivial cases.

The TFT interpretation in which the puncture operator is T_{-1}, instead of T_1, is perfectly specular (due to the $\mathbf{Z_2}$ symmetry of the underlying Toda lattice hierarchy under the exchange of the left with the right sector) and there is no need to describe it in detail here. We call the corresponding TFT coupled to topological gravity $\mathcal{T}_{-1}$. The physical nature of the symmetry between left and right sector of the two-matrix model is not clear. It might be related to some duality symmetry of the underlying string theory.

[§] Eq.(2.12) is true if we exclude the exceptional case $\chi_{r,s} \neq T_{-1}$, because we would have in the RHS $< QQ >= \ln x$

2.2 Another TFT interpretation: puncture operator Q.

In this interpretation the puncture operator is $Q \equiv T_0$. This is motivated by the fact that, according to the Penner model (see section 1.1), Q represents a puncture on a Riemann surface. A priori however there is no compelling reason why this interpretation should work as the previous one, based on the analogy with the $c < 1$ TFT models. The Penner model concerns the virtual Euler characteristic and involves only correlators of Q. Nevertheless the interpretation turns out to work.

Actually this interpretation has been already introduced in ref.[4]. For the sake of completeness, we review it here and whenever necessary, we complete the description given there. The set of primary fields is the same as in $\mathcal{T}_0$, i.e. $T_n \equiv \chi_{n,0}$, $T_{-n} \equiv \chi_{0,n}$, and $T_0 \equiv Q$, while all the other $\chi_{n,m}$ are descendants.

The metric is given by

$$\eta_{a,b} =< QT_aT_b > \tag{2.14}$$

where a and b are integers. The only nonzero elements are

$$\eta_{n,-n} = \eta_{-n,n} =< QT_nT_{-n} >\equiv \frac{\partial}{\partial x} < \chi_{n,0}\chi_{0,n} >= n^2x^{n-1}, \quad \eta_{0,0} = x^{-1}$$

This metric is non–degenerate, the inverse is $\eta^{k,l}$ with

$$\eta^{n,-n} = \eta^{-n,n} = n^{-2}x^{-n+1}, \qquad \eta^{0,0} = x$$

while all the other elements vanish. The associativity condition is easily seen to be satisfied since the only nonvanishing three–point functions among primaries are

$$\begin{aligned} C_{n,m,-n-m} &= C_{-n,-m,n+m} =< T_{-n}T_{-m}T_{n+m} >= nm(n+m)x^{n+m-1} \\ C_{n,-m,m-n} &= \begin{cases} nm(n-m)x^{n-1}, & n > m \\ nm(m-n)x^{m-1}, & n < m \end{cases} \end{aligned} \tag{2.15}$$

beside $C_{0,n,m} \equiv \eta_{n,-n}\delta_{n+m,0}$. As is easy to prove, the primary fields form the commutative associative algebra $\mathcal{A}_0$

$$T_aT_b = \sum_c C_{a,b}{}^cT_c, \qquad C_{a,b}{}^c \equiv \sum_d C_{a,b,d}\eta^{d,c}$$

where again T_0 is identified with the identity in $\mathcal{A}_0$.

The *recursion relations* in $\mathcal{S}_0$ are

$$< \chi_{r,s}\chi_{r_1,s_1}\chi_{r_2,s_2} >= M(r,s)\sum_{l,k} < \chi_{r-1,s-1}T_a > \eta^{a,b} < T_b\chi_{r_1,s_1}\chi_{r_2,s_2} > \tag{2.16}$$

The proof is very simple and can be found in [4]. [¶]

The *puncture equations* are designed to connect the CF's of of the type

$$< Q\chi_{r_1,s_1}\chi_{r_2,s_2}\cdots\chi_{r_n,s_n} >,$$

[¶]Here again the relation (2.16) does not work when $r = s = 1$. This is an exceptional case due to the fact that on the RHS there appears the correlator $< QQ >= \ln x$.

where the χ's are extra states, with CF's including neighboring ascendants of them. For dimensional reason the latter can only be $< \chi_{r_1,s_1} \dots \chi_{r_i-1,s_i-1} \dots \chi_{r_n,s_n} >$. For the CF's (1.4) we have

$$
\begin{aligned}
&< Q\chi_{r_1,s_1}\chi_{r_2,s_2} \dots \chi_{r_n,s_n} >= \\
&\qquad \sum_{i=1}^{n} \frac{M(r_i,s_i)}{M(r_i-1,s_i-1)} \frac{\Sigma-1}{n} < \chi_{r_1,s_1} \dots \chi_{r_i-1,s_i-1} \dots \chi_{r_n,s_n} >
\end{aligned}
\tag{2.17}
$$

where $\Sigma = r_1 + \dots + r_n = s_1 + \dots + s_n$. In fact the LHS is

$$
< Q\chi_{r_1,s_1}\chi_{r_2,s_2} \dots \chi_{r_n,s_n} >= x^{\Sigma-n+1} M(r_1,s_1) \dots M(r_n,s_n)(\Sigma-1)\dots(\Sigma-n+2)
$$

On the other hand the generic term in the RHS of (2.17) contains

$$
\begin{aligned}
< \chi_{r_1,s_1} \dots \chi_{r_k-1,s_k-1} \dots \chi_{r_n,s_n} > \;=\; & M(r_1,s_1)\dots M(r_k-1,s_k-1)\dots M(r_n,s_n)\, \cdot \\
& \cdot (\Sigma-2)\dots(\Sigma-n+2)x^{\Sigma-n+1}
\end{aligned}
$$

Summing all the contributions in the RHS of (2.17) we obtain the equality with the LHS. However a relation similar to (2.17) holds in general. In fact in genus zero the LHS of (2.17) is nothing but the derivative of

$$
< \chi_{r_1,s_1}\chi_{r_2,s_2} \dots \chi_{r_n,s_n} >
\tag{2.18}
$$

with respect to x. On the other hand

$$
< \chi_{r_1,s_1} \dots \chi_{r_i-1,s_i-1} \dots \chi_{r_n,s_n} >
\tag{2.19}
$$

is also the derivative of (2.18) with respect to x up to a multiplicative rational factor. Therefore by taking a suitable combination of all the (2.19), we can certainly reproduce the LHS of (2.17).

In summary, in this TFT interpretation $Q \equiv T_0$ *is the puncture operator,* T_n *and* T_{-n} *(*n *positive) are the primary fields, while* $\chi_{n+k,k}$ *and* $\chi_{k,n+k}$*, with* k *positive, are, respectively, the descendants. In particular* $\chi_{k,k}$ *are the descendants of* Q*. Once again we have to do with an unperturbed TFT coupled to topological gravity. Let us call it* $\mathcal{T}_0$*.*

It is evident that the puncture equation is determined by the dispersionless flow in x, i.e. in N, the size of the matrices in the two–matrix model. Therefore it does not extend, as it is, to higher genus. For example, for one point functions, the all–genus puncture equation becomes

$$
< (1-e^{-Q})\chi_{r,r} >_{\text{all-genus}} = r < \chi_{r-1,r-1} >_{\text{all-genus}}
\tag{2.20}
$$

This is exact and is clearly the generalization of (2.17) to every genus.

The rule is very simple: to generalize (2.17) one has simply to write down the exact flow in N*.* The latter is provided by the two–matrix model. In general we can repeat the same conclusion as in the previous subsection: *the higher genus puncture and recursion relations are nothing but the* W *constraints of the two matrix model.*

For the $< Q^n >$ correlators the same remarks holds as in the previous subsection.

We showed in [4] that a LG interpretation of $\mathcal{T}_0$ can be introduced, but we will not insist here on such interpretation.

3 Perturbation of the TFT's.

We study now the TFT's $\mathcal{T}_0$ and $\mathcal{T}_1$ under the most general infinitesimal tachyonic perturbation. Since in the infinitesimal case, the perturbation by an operator T_n or T_{-n} appears linearly, it is enough to consider such perturbations one by one. Let us study hereby the perturbations by T_p and by T_{-q}, where p and q are positive integers.

The relevant correlators (one, two and three–point functions) perturbed by means of T_1 and T_{-1} have been given in [4]. The ones perturbed by means of T_2 and T_{-2} have been given in [7]. Finally the correlators infinitesimally perturbed by T_p and T_{-q} can be obtained from the formulas of the following sections or by solving the coupling conditions in the two matrix model.

Below we give the results relevant to the subsequent developments. Let us set

$$pt_{1,p} = \epsilon_p \ll 1, \qquad qt_{2,q} = \zeta_q \ll 1$$

Then we have

$$\begin{aligned}
C_{n,m,l} &= nml(q-1)\zeta_q x^{q-2}\delta_{n+m+l,q} \\
C_{n,m,-l} &= nm(n+m)x^{n+m-1}\delta_{l,n+m} + \epsilon_p nml(l-1)x^{l-2}\delta_{l,n+m+p} + \\
&\quad \zeta_q nml\Big((l-1)\theta(l-m)\theta(l-n) + (n-1)\theta(n-l)\theta(l-m) + \\
&\quad (m-1)\theta(m-l)\theta(l-n) + (q-1)\theta(n-l)\theta(m-l)\Big)x^{l+q-2}\delta_{l+q,n+m} \\
C_{n,-m,-l} &= nmlx^{n-1}\delta_{n,m+l} + \epsilon_p nml\Big((n-1)\theta(n-m)\theta(n-l) + (m-1)\theta(m-n)\theta(n-l) + \\
&\quad (l-1)\theta(n-m)\theta(l-n) + (p-1)\theta(l-n)\theta(m-n)\Big)x^{p+n-2}\delta_{n+p,m+l} + \\
&\quad \zeta_q nml(n-1)x^{n-2}\delta_{n,m+l+q} \\
C_{-n,-m,-l} &= \epsilon_p nml(p-1)x^{p-2}\delta_{p,n+m+l}
\end{aligned} \tag{3.21}$$

We have moreover

$$\begin{aligned}
C_{0,n,m} &= \zeta_q nm(q-1)x^{q-2}\delta_{q,n+m} \\
C_{0,n,-m} &= n^2x^{n-1}\delta_{n,m} + \epsilon_p nm(m-1)x^{m-2}\delta_{m,n+p} + \zeta_q nm(n-1)x^{n-2}\delta_{m+q,n} \\
C_{0,-n,-m} &= \epsilon_p nm(p-1)x^{p-2}\delta_{p,n+m} \\
C_{0,0,n} &= \zeta_q q(q-1)x^{q-2}\delta_{n,q} \\
C_{0,0,-n} &= \epsilon_p p(p-1)x^{p-2}\delta_{n,p} \\
C_{0,0,0} &= x^{-1}
\end{aligned} \tag{3.22}$$

where

$$\theta(n) = \begin{cases} 1, & n > 0 \\ 1/2, & n = 0 \\ 0, & n < 0 \end{cases}$$

As a consequence of the perturbation the metrics and structure constants of the TFT's are modified. We are going to see next that all the the topological field theory properties are nevertheless satisfied.

3.1 T_1 puncture operator: perturbation

In order to show that the $c = 1$ string theory with puncture operator T_1 perturbed as above is a TFT, we will prove that the inverse metric exists and that the associativity conditions are satisfied. We prove everything to the first order in ϵ_p and ζ_q, which are infinitesimal. The metric is given by $\eta_{a,b} =< T_1 T_a T_b >$. Its non–vanishing elements are therefore

$$\begin{aligned}
\eta_{n,-n-1} &= n(n+1)x^n, \qquad \eta_{0,-1} = 1 \\
\eta_{n,-n-p-1} &= \epsilon_p n(n+p)(n+p+1)x^{n+p-1}, \\
\eta_{-n,n-p-1} &= \epsilon_p n(p-1)(p+1-n)x^{p-1}, \qquad n < p+1 \\
\eta_{0,-p-1} &= \epsilon_p p(p+1)x^{p-1}, \qquad \eta_{0,q-1} = \zeta_q(q-1)^2 x^{q-2} \\
\eta_{n,-n+q-1} &= \zeta_q n(n-1)(n-q+1)x^{n-1}, \qquad n > q-1 \\
\eta_{n,q-n-1} &= n(q-1)(q-n-1)x^{q-2}, \qquad n < q-1
\end{aligned} \tag{3.23}$$

plus the ones which are obtained from these via the symmetry $\eta_{a,b} = \eta_{b,a}$.

The inverse metric exists, its nonvanishing elements are

$$\begin{aligned}
\eta^{n,-n-1} &= \frac{x^{-n}}{n(n+1)}, \qquad \eta^{0,-1} = 1 \\
\eta^{n,-n+p-1} &= -\epsilon_p \frac{x^{-n+p-1}}{n-p+1}, \qquad n > p-1 \\
\eta^{n,p-n-1} &= -\epsilon_p \frac{p-1}{n(p-n-1)}, \qquad n < p-1 \\
\eta^{0,p-1} &= -\epsilon_p, \qquad \eta^{0,-q-1} = -\zeta_q \frac{q-1}{q+1} \\
\eta^{n,-n-q-1} &= -\zeta_q \frac{n+q-1}{n(n+q+1)} x^{-1-n} \\
\eta^{-n,n-q-1} &= -\zeta_q \frac{q-1}{n(q-n+1)} x^{-1}, \qquad n < q+1
\end{aligned} \tag{3.24}$$

plus the ones that can be obtained from them via the symmetry $\eta^{a,b} = \eta^{b,a}$.

One way to prove the associativity conditions for a perturbation ϵ_p is to verify that

$$\begin{aligned}
C_{n,m,a}\eta^{a,b}C_{b,k,-l} &= C_{n,k,a}\eta^{a,b}C_{b,m,-l} \\
C_{-n,-m,a}\eta^{a,b}C_{b,-k,l} &= C_{-n,-k,a}\eta^{a,b}C_{b,-m,l} \\
C_{n,m,a}\eta^{a,b}C_{b,-k,-l} &= C_{n,-k,a}\eta^{a,b}C_{b,m,-l} \\
C_{-n,-m,a}\eta^{a,b}C_{b,-k,-l} &= C_{-n,-k,a}\eta^{a,b}C_{b,-m,-l}
\end{aligned}$$

are identities up to the first order in ϵ_p. This is a lengthy but straightforward exercise on the basis of (3.21,3.22, 3.27). These four identities are enough since all the other identities that appear in (2.6) can be obtained from them either by using the symmetry properties of C and η, or, when some of the indices n, m, k, l are replaced by 0, by remarking that we have formally

$$C_{0,a,b} = \lim_{n\to 0} \frac{C_{n,a,b}}{n}, \qquad C_{0,0,b} = \lim_{n,m\to 0} \frac{C_{n,m,b}}{nm}$$

We proceed in a similar way with a ζ_q perturbation.

The metric (3.23) depends in general on the perturbation parameters, but it is easy to find redefinitions of the primaries so as to recover a constant metric. For example, for a ϵ_p perturbation, we define

$$
\begin{aligned}
&\hat{T}_n = T_n - \epsilon_p \frac{n(n+\frac{p-1}{2})}{n+p} x^{-1} T_{n+p} \\
&\hat{T}_{-n} = T_{-n} - \epsilon_p \frac{1}{4}\frac{n(p-3)}{p-n} x^{n-1} T_{p-n}, \qquad n<p \\
&\hat{T}_{-p} = T_{-p} - \epsilon_p \frac{1}{4} p(p-3) x^{p-1} T_0 \\
&\hat{T}_{-n} = T_{-n}, \qquad n>p \\
&\hat{T}_0 = T_0 - \epsilon_p \frac{x^{-1}}{2}\frac{p-1}{p} T_p
\end{aligned}
\tag{3.25}
$$

In terms of the hatted fields the metric becomes constant and equal to the unperturbed one. A similar redefinition can be done for any ζ_q perturbation.

As for the coupling to topological gravity, the puncture equations and recursion relations have to be, in general, suitably modified with respect to the previous section. There is however no point in writing them down explicitly. They are nothing but particular aspects of the W–constraints of the two–matrix model.

The situation with T_{-1} as puncture operator is exactly specular.

3.2 Q puncture operator: perturbation

We now do the same when Q is the puncture operator. The metric is given by $\eta_{a,b} = < QT_aT_b >$. The only non–vanishing metric elements are therefore

$$
\begin{aligned}
\eta_{n,-n} &= n^2 x^{n-1}, \qquad \eta_{0,0} = x^{-1} \\
\eta_{n,-n-p} &= \epsilon_p n(n+p)(n+p-1) x^{n+p-2}, \\
\eta_{-n,n-p} &= \epsilon_p n(p-1)(p-n) x^{p-2}, \qquad n<p \\
\eta_{0,-p} &= \epsilon_p p(p-1) x^{p-2}, \qquad \eta_{0,q} = \zeta_q q(q-1) x^{q-2} \\
\eta_{n,-n+q} &= \zeta_q n(n-1)(n-q) x^{n-2}, \qquad n>q \\
\eta_{n,q-n} &= \zeta_q n(q-1)(q-n) x^{q-2}, \qquad n<q
\end{aligned}
\tag{3.26}
$$

plus the ones which are obtained from these via the symmetry $\eta_{a,b} = \eta_{b,a}$.

The inverse metric exists, its nonvanishing elements are

$$
\begin{aligned}
\eta^{n,-n} &= \frac{x^{1-n}}{n^2}, \qquad \eta^{0,0} = x \\
\eta^{n,-n+p} &= -\epsilon_p \frac{n-1}{n(n-p)} x^{-n+p}, \qquad n>p \\
\eta^{n,p-n} &= -\epsilon_p \frac{p-1}{n(p-n)}, \qquad n<p \\
\eta^{0,p} &= -\epsilon_p \frac{p-1}{p}, \qquad \eta^{0,-q} = -\zeta_q \frac{q-1}{q}
\end{aligned}
\tag{3.27}
$$

$$
\begin{aligned}
\eta^{n,-n-q} &= -\zeta_q \frac{n+q-1}{n(n+q)} x^{-n} \\
\eta^{-n,n-q} &= -\zeta_q \frac{q-1}{n(q-n)}, \qquad\qquad n<q
\end{aligned}
$$

plus the ones that can be obtained from them via the symmetry $\eta^{a,b} = \eta^{b,a}$.

Using these formulas one can prove the associativity conditions in the same way as above.

The metric (3.26) depends in general on the perturbation parameters. However it is easy to find redefinitions of the primaries so as to recover a constant metric. For example, for a ϵ_p perturbation, we define

$$
\begin{aligned}
&\hat{T}_n = T_n - \epsilon_p \frac{n(n+\frac{p-1}{2})}{n+p} x^{-1} T_{n+p} \\
&\hat{T}_{-n} = T_{-n} - \epsilon_p \frac{1}{4}\frac{n(p-1)}{p-n} x^{n-1} T_{p-n}, \qquad n<p \\
&\hat{T}_{-n} = T_{-n}, \qquad n \geq p \\
&\hat{T}_0 = T_0 - \epsilon_p \frac{x^{-1}}{2}\frac{p-1}{p} T_p
\end{aligned}
\tag{3.28}
$$

In terms of the hatted fields the metric becomes constant and equal to the unperturbed one. A similar redefinition can be done for any ζ_q perturbation.

4 The $c = 1$ string theory and the extended Toda lattice hierarchy

Let us now turn our attention to the Toda lattice hierarchy. This allows us on one the hand to compute correlators when finite perturbations are switched on – in particular one can derive the results used in the previous sections. On the other hand we prepare the ground to introduce, in the following section, a LG representation of $\mathcal{T}_1$. We recall that in the extended (restricted) Toda lattice hierarchy the extra states are admitted (excluded).

We have already pointed out that the $c = 1$ string theory is described at the self–dual point by the extended two matrix model, which is equivalent to the extended $2d$ Toda lattice hierarchy subject to the coupling conditions. In this section we will only pay attention to the genus zero case, therefore we will briefly review the dispersionless extended Toda lattice hierarchy, and its restriction to the $c = 1$ string theory.

4.1 Dispersionless extended Toda lattice hierarchy

The dispersionless extended Toda hierarchy is based on four objects, which are Laurent series in the complex variable ζ. Two of them are the so–called Lax operators

$$
L = \zeta + \sum_{l=0}^{\infty} a_l \zeta^{-l}, \qquad \widetilde{L} = \frac{R}{\zeta} + \sum_{l=0}^{\infty} \frac{b_l}{R^l} \zeta^l, \tag{4.1}
$$

The other two are

$$
M = \sum_{r=1}^{\infty} r t_{1,r} L^{r-1} + x L^{-1} + \sum_{r=1}^{\infty} \frac{\partial F}{\partial t_{1,r}} L^{-r-1}, \tag{4.2}
$$

$$\hat{\sigma}(\widetilde{M}) = \sum_{r=1}^{\infty} r t_{2,r} \widetilde{L}^{r-1} + x\widetilde{L}^{-1} + \sum_{r=1}^{\infty} \frac{\partial F}{\partial t_{2,r}} \widetilde{L}^{-r-1}. \tag{4.3}$$

where the operation $\hat{\sigma}$ is defined as follows

$$\hat{\sigma}(\zeta) = \frac{R}{\zeta}, \qquad \hat{\sigma}(f) = f, \quad \forall \text{ function } \ f. \tag{4.4}$$

In these equations, F is connected with the τ–function (see below) and will be subsequently interpreted as the free energy. R, a_l and b_l are 'fields' (i.e. functions of the couplings). With respect to the basic Poisson bracket

$$\{\zeta, \quad x\} = \zeta, \tag{4.5}$$

the four Laurent series given above satisfy the fundamental relations

$$\{L, \quad M\} = 1, \qquad \{\hat{\sigma}(\widetilde{L}), \quad \widetilde{M}\} = 1. \tag{4.6}$$

The dispersionless extended Toda hierarchy can be represented in several different ways.

<u>*The first representation*</u> :

The dispersionless extended Toda hierarchy can be written as follows

$$\frac{\partial L}{\partial g_{r,s}} = \{L, \quad (L^r \widetilde{L}^s)_-\}, \tag{4.7a}$$

$$\frac{\partial \widetilde{L}}{\partial g_{r,s}} = \{(L^r \widetilde{L}^s)_+, \quad \widetilde{L}\}, \tag{4.7b}$$

where $g_{r,s}$ are the flow parameters or coupling constants with non–negative integers (r, s) (they are not simutaneously zero) introduced in section 1. For any Laurent series $f(\zeta) = \sum_i f_i \zeta^i$, we denote

$$f_+(\zeta) = \sum_{i \geq 0} f_i \zeta^i, \qquad f_-(\zeta) = \sum_{i<0} f_i \zeta^i,$$

and

$$f_{\geq k}(\zeta) = \sum_{i \geq k} f_i \zeta^i, \qquad f_{\leq l}(\zeta) = \sum_{i \leq l} f_i \zeta^i, \qquad f_{(k)}(\zeta) = f_k.$$

For the sake of completeness, one may add to eqs.(4.7a,4.7b) two x–flow equations

$$\frac{\partial L}{\partial x} = \{(\ln L)_+, \quad L\}, \qquad \frac{\partial \widetilde{L}}{\partial x} = \{(\ln \widetilde{L})_+, \quad \widetilde{L}\}. \tag{4.8}$$

The τ-*function* of the above integrable hierarchy (denoted by e^F) is linked to the fields a_l and b_l thorough the following relation

$$\frac{\partial}{\partial g_{r,s}} F = \int_0^x (L^r \widetilde{L}^s)_{(0)}(y) dy. \tag{4.9}$$

This relation, together with the flow equations, leads to

$$\frac{\partial^2}{\partial t_{1,1}\partial t_{2,1}}\ln R=\partial_x^2 R, \tag{4.10}$$

which is the continuum version of the $2d$ Toda lattice equation. One may also re–express the dispersionless extended Toda hierarchy in terms of M and $\widetilde{M}$

$$\frac{\partial M}{\partial g_{r,0}}=\{L_+^r,\quad M\},\qquad r\geq 1 \tag{4.11a}$$

$$\frac{\partial M}{\partial g_{r,s}}=\{M,\quad (L^r\widetilde{L}^s)_-\},\qquad r\geq 0,\quad s\geq 1 \tag{4.11b}$$

$$\frac{\partial\hat{\sigma}(\widetilde{M})}{\partial g_{0,s}}=\{\hat{\sigma}(\widetilde{M}),\quad \widetilde{L}_-^s\},\qquad s\geq 1 \tag{4.11c}$$

$$\frac{\partial\hat{\sigma}(\widetilde{M})}{\partial g_{r,s}}=\{(L^r\widetilde{L}^s)_+,\quad \hat{\sigma}(\widetilde{M})\},\qquad r\geq 1,\quad s\geq 0. \tag{4.11d}$$

Both representations (4.7a,4.7b) and (4.11a–4.11d) will be useful in our later discussion.

It is useful to express the Toda lattice in terms of the underlying linear systems, i.e. by means of suitable *Baker–Akhiezer functions*. The Baker–Akhiezer functions $\Psi(\lambda_1)$ and $\Psi(\lambda_2)$, appropriate for our case, are given by

$$\ln\Psi(\lambda_1)=\sum_{r=1}^{\infty}t_{1,r}\lambda_1^r+x\ln\lambda_1-\sum_{r=1}^{\infty}\frac{1}{r\lambda_1^r}\frac{\partial F}{\partial t_{1,r}}, \tag{4.12}$$

$$\ln\Phi(\lambda_2)=\sum_{r=1}^{\infty}t_{2,r}\lambda_2^r+x\ln\lambda_2-\sum_{r=1}^{\infty}\frac{1}{r\lambda_2^r}\frac{\partial F}{\partial t_{2,r}}. \tag{4.13}$$

The spectral parameters λ_1,λ_2 and the Lax operators $L,\hat{\sigma}(\widetilde{L})$ are interchangeable, so we have

$$M(L)=\frac{d\ln\Psi(L)}{dL},\qquad\qquad \widetilde{M}(\widetilde{L})=\frac{d\ln\Phi(\hat{\sigma}(\widetilde{L}))}{d\hat{\sigma}(\widetilde{L})}. \tag{4.14}$$

The equations of motion of Baker–Akhiezer functions are

$$\frac{\partial\ln\Psi(L)}{\partial x}=\ln\zeta,\qquad \frac{\partial\ln\Phi(\hat{\sigma}(\widetilde{L}))}{\partial x}=\ln\zeta, \tag{4.15a}$$

$$\frac{\partial\ln\Psi(L)}{\partial g_{r,0}}=L_+^r(\zeta),\qquad r\geq 1; \tag{4.15b}$$

$$\frac{\partial\ln\Psi(L)}{\partial g_{r,s}}=-(L^r\widetilde{L}^s)_-(\zeta),\qquad r\geq 0,\quad s\geq 1; \tag{4.15c}$$

$$\frac{\partial\ln\Phi(\hat{\sigma}(\widetilde{L}))}{\partial g_{0,s}}=\Big(\hat{\sigma}(\widetilde{L}^s)\Big)_+(\zeta),\qquad s\geq 1; \tag{4.15d}$$

$$\frac{\partial\ln\Phi(\hat{\sigma}(\widetilde{L}))}{\partial g_{r,s}}=-\Big(\hat{\sigma}(L^r\widetilde{L}^s)\Big)_-(\zeta),\qquad r\geq 1,\quad s\geq 0. \tag{4.15e}$$

where the arguments in the brackets on the LHS are fixed when taking derivatives with respect to the coupling parameters. These equations, on one hand reproduce eqs.(4.11a–4.11d), on

the other hand lead (together with eqs.(4.12,4.13)) to

$$(L^r\widetilde{L}^s)_- = \sum_{k=1}^{\infty}\frac{1}{kL^k}\frac{\partial^2 F}{\partial g_{k,0}\partial g_{r,s}}, \tag{4.16a}$$

$$(L^r\widetilde{L}^s)_{\geq 1} = \sum_{k=1}^{\infty}\frac{1}{k\widetilde{L}^k}\frac{\partial^2 F}{\partial g_{0,k}\partial g_{r,s}}, \tag{4.16b}$$

where r, s are non–negative integers (not simutaneously zero). In the case $(r, s) = (0, 0)$, we have

$$\ln\zeta = \ln L - \sum_{k=1}^{\infty}\frac{1}{kL^k}\frac{\partial^2 F}{\partial g_{k,0}\partial x}, \tag{4.17a}$$

$$\ln\zeta = \ln\hat{\sigma}(\widetilde{L}) - \sum_{k=1}^{\infty}\frac{1}{k\hat{\sigma}(\widetilde{L}^k)}\frac{\partial^2 F}{\partial g_{0,k}\partial x}. \tag{4.17b}$$

Combining eqs.(4.16a–4.17b), we are able to obtain the following identities

$$L^r\widetilde{L}^s = \sum_{k=1}^{\infty}\frac{1}{kL^k}\frac{\partial^2 F}{\partial g_{k,0}\partial g_{r,s}} + \frac{\partial^2 F}{\partial x\partial g_{r,s}} + \sum_{k=1}^{\infty}\frac{1}{k\widetilde{L}^k}\frac{\partial^2 F}{\partial g_{0,k}\partial g_{r,s}}, \tag{4.18}$$

where $(r, s) \neq (0, 0)$, and

$$\ln(L\widetilde{L}) = \sum_{k=1}^{\infty}\frac{1}{kL^k}\frac{\partial^2 F}{\partial g_{k,0}\partial x} + \ln R + \sum_{k=1}^{\infty}\frac{1}{k\widetilde{L}^k}\frac{\partial^2 F}{\partial g_{0,k}\partial x}. \tag{4.19}$$

On the other hand, eq.(4.16a) immediately leads to

$$\frac{\partial^2 F}{\partial g_{k,0}\partial g_{r,s}} = \oint (L^r\widetilde{L}^s)_- dL^k, \tag{4.20}$$

similarly

$$\frac{\partial^2 F}{\partial g_{0,k}\partial g_{r,s}} = \oint (L^r\widetilde{L}^s)_{\geq 1} d\widetilde{L}^k. \tag{4.21}$$

We remark that these formulas are valid in general, without any restriction on the couplings, and even before trucating to $c = 1$ string theory, as will be done in the following subsection.

The second representation :

Equivalently we may represent the integrable hierarchy in terms of two conjugate pairs (L, M) and $(\widetilde{L}, \widetilde{M})$ and infinite many Poisson brackets. From eqs.(4.7a, 4.7b), and eqs.(4.11a–4.11d), as well as eqs.(4.6), we can derive

$$\frac{d}{d\zeta}\Big(L^i(\zeta)\Big)_+ = \frac{\partial M(\zeta)}{\partial t_i}\frac{\partial L}{\partial\zeta} - \frac{\partial L(\zeta)}{\partial t_i}\frac{\partial M}{\partial\zeta}, \tag{4.22a}$$

$$\frac{d}{d\zeta}\Big((L^i\widetilde{L}^j)(\zeta)\Big)_- = \frac{\partial L(\zeta)}{\partial g_{i,j}}\frac{\partial M}{\partial\zeta} - \frac{\partial M(\zeta)}{\partial g_{i,j}}\frac{\partial L}{\partial\zeta}, \qquad i \geq 0, j \geq 1, \tag{4.22b}$$

$$\frac{d}{d\zeta}\Big(\hat{\sigma}(\widetilde{L}^j)(\zeta)\Big)_+ = \frac{\partial\widetilde{M}(\zeta)}{\partial\widetilde{t}_j}\frac{\partial\hat{\sigma}(\widetilde{L})}{\partial\zeta} - \frac{\partial\hat{\sigma}(\widetilde{L})(\zeta)}{\partial\widetilde{t}_j}\frac{\partial\widetilde{M}}{\partial\zeta}, \tag{4.22c}$$

$$\frac{d}{d\zeta}\Big(\hat{\sigma}(L^i\widetilde{L}^j)(\zeta)\Big)_- = \frac{\partial\hat{\sigma}(\widetilde{L})(\zeta)}{\partial g_{i,j}}\frac{\partial\widetilde{M}}{\partial\zeta} - \frac{\partial\widetilde{M}(\zeta)}{\partial g_{i,j}}\frac{\partial\hat{\sigma}(\widetilde{L})}{\partial\zeta}, \qquad i \geq 1, j \geq 0. \tag{4.22d}$$

These equations have a Poisson bracket structure which can be made explicit by introducing one Poisson bracket for each coupling constant

$$\{\zeta, \quad g_{i,j}\}_{i\otimes j} = \zeta. \tag{4.23}$$

In terms of these Poisson brackets, the above flow equations can be rewritten as

$$\zeta\frac{d}{d\zeta}\Big(L^i(\zeta)\Big)_+ \quad = \quad \{L(\zeta), \quad M(\zeta)\}_{i\otimes 0}, \tag{4.24a}$$

$$-\zeta\frac{d}{d\zeta}\Big((L^i\widetilde{L}^j)(\zeta)\Big)_- \quad = \quad \{L(\zeta), \quad M(\zeta)\}_{i\otimes j}, \qquad i \geq 0, j \geq 1, \tag{4.24b}$$

$$\zeta\frac{d}{d\zeta}\Big(\hat{\sigma}(\widetilde{L}^j)(\zeta)\Big)_+ \quad = \quad \{\hat{\sigma}(\widetilde{L})(\zeta), \quad \widetilde{M}(\zeta)\}_{0\otimes j}, \tag{4.24c}$$

$$-\zeta\frac{d}{d\zeta}\Big(\hat{\sigma}(L^i\widetilde{L}^j)(\zeta)\Big)_- \quad = \quad \{\hat{\sigma}(\widetilde{L})(\zeta), \quad \widetilde{M}(\zeta)\}_{i\otimes j}, \qquad i \geq 1, j \geq 0. \tag{4.24d}$$

This is our second representation of the extended dispersionless Toda hierarchy. The first description is a good framework to describe Hamiltonian structures. The second representation will naturally lead to the Landau–Ginzburg formulation.

Actually there other possible formulations of the hierarchy. For example after introducing the *canonical* momentum $p \equiv \zeta + a_0$, we can reformulate the hierarchy in terms of the canonical Poisson bracket

$$\{p, \quad t_{1,1}\}_{\mathrm{KP}} = 1,$$

instead of using (4.5). The resulting form of the hierarchy is nothing but the extension of the standard dispersionless KP hierarchy. The flows corresponding to discrete states are related to additional symmetries of the KP hierarchy.

4.2 Truncation to the $c = 1$ string theory

Let us introduce some notations

$$< \chi_{i_1,j_1}\chi_{i_2,j_2}\dots\chi_{i_n,j_n} >\equiv \frac{\partial^n F}{\partial g_{i_1,j_1}\partial g_{i_2,j_2}\dots\partial g_{i_n,j_n}}, \qquad < Q >\equiv \frac{\partial F}{\partial x}.$$

Beside $\mathcal{S}_0$, we define three more particular subspaces of the full coupling space

$$\begin{aligned} \mathcal{S} &= \{t_{1,r}, \quad t_{2,s}, \quad \forall r,s \geq 1, \quad x, \quad g_{1,1} = -1\}; \\ \mathcal{S}_+ &= \{t_{1,r}, \quad \forall r \geq 1, \quad t_{2,1}, \quad x, \quad g_{1,1} = -1\}; \\ \mathcal{S}_- &= \{t_{1,1}, \quad t_{2,s}, \quad \forall s \geq 1, \quad x, \quad g_{1,1} = -1\}; \end{aligned}$$

The number of couplings in $\mathcal{S}, \mathcal{S}_+$ and $\mathcal{S}_-$ is understood to be arbitrarily large but finite, therefore $\mathcal{S}$ denotes any model $\mathcal{M}_{p,q}$, $\mathcal{S}_+$ any model $\mathcal{M}_{p,1}$ and $\mathcal{S}_-$ any model $\mathcal{M}_{1,q}$, evaluated at $g_{1,1} = -1$.

The extended Toda lattice hierarchy provides the description of $c = 1$ string theory, *if and only if* we impose certain constraints, the coupling conditions. In the dispersionless limit, the fundamental constraints (or *coupling conditions*) take the following simple form

$$M + \sum_{r,s\geq 1} r g_{r,s} L^{r-1}\widetilde{L}^s = 0, \qquad \hat{\sigma}(\widetilde{M}) + \sum_{r,s\geq 1} s g_{r,s} L^r \widetilde{L}^{s-1} = 0. \tag{4.25}$$

After this restriction, the dispersionless τ–function F coincides with the genus zero free energy of the $c = 1$ string theory. Therefore, $\chi_{i,j}(i, j \geq 1)$ represent the discrete states, and T_i, T_{-j} are tachyons, while Q denotes the cosmological operator.

The coupling conditions (4.25) together with the integrable hierarchy lead to the dispersionless $W_{1+\infty}$ constraints acting on the free energy. In order to write them down in a compact way, we define

$$T_n^{[m]}(1) = \sum_{\substack{i_1,\ldots,i_m \geq 1 \\ j_1,\ldots,j_m \geq 1}} i_1 \ldots i_m g_{i_1,j_1} \ldots g_{i_m,j_m} \frac{\partial}{\partial g_{i_1+\ldots+i_m+n-m,j_1+\ldots+j_m}}, \quad n \geq 0, m \geq 1. \quad (4.26)$$

They satisfy the algebra

$$[T_n^{[m]}(1), T_r^{[s]}(1)] = (sn - rm) T_{n+r}^{[m+s-1]}(1), \qquad m, s \geq 1; \quad n, r \geq 0. \quad (4.27)$$

This is nothing but the Lie algebra the area–preserving diffeomorphisms. There is another set of generators

$$T_n^{[m]}(2) = \sum_{\substack{i_1,\ldots,i_m \geq 1 \\ j_1,\ldots,j_m \geq 1}} j_1 \ldots j_m g_{i_1,j_1} \ldots g_{i_m,j_m} \frac{\partial}{\partial g_{i_1+\ldots+i_m,j_1+\ldots+j_m+n-m}}, \quad n \geq 0, m \geq 1. \quad (4.28)$$

They form another area–preserving diffeomorphism algebra. In terms of these operators, the constraints can be written as

$$T_n^{[m]}(1) F = \frac{(-1)^m}{(n+1)(m+1)} \mathrm{res}_\zeta \Big(M^{m+1}(L) dL^{n+1} \Big), \quad m \geq 1, \quad n \geq 0. \quad (4.29)$$

and

$$T_n^{[m]}(2) F = \frac{(-1)^m}{(n+1)(m+1)} \mathrm{res}_\zeta \Big(\hat{\sigma}(M^{m+1})(\widetilde{L}) d\widetilde{L}^{n+1} \Big), \quad m \geq 1, \quad n \geq 0. \quad (4.30)$$

The simplest case $(n = 0, m = 1)$ is of particular importance, we may write it explicitly as follows

$$\frac{\partial}{\partial t_{2,1}} F = \sum_{m,j \geq 0, (m,j) \neq (0,0)} \Big((m+1) g_{m+1,j} + x t_{1,1} + \delta_{m,0} \delta_{j,1} \Big) \frac{\partial}{\partial g_{m,j}} F. \quad (4.31)$$

Similarly, we have

$$\frac{\partial}{\partial t_{1,1}} F = \sum_{i,n \geq 0, (i,n) \neq (0,0)} \Big((n+1) g_{i,n+1} + x t_{2,1} + \delta_{i,1} \delta_{n,0} \Big) \frac{\partial}{\partial g_{i,n}} F. \quad (4.32)$$

Eq.(4.31) and eq.(4.32) are the two simplest constraints, both of them have the structure of the string equation. So we can choose either eq.(4.31), or eq.(4.32) as string equation. Thus we have two possible ways to specify the puncture operator, the primary fields, and the gravitational descendants. This is but another manifestation of the duality between the T_1 and T_{-1} picture which we have already found in section 3. In order to see this more closely, let us consider a general n–point function in $\mathcal{S}_+$,

$$< \chi_{i_1,j_1} \chi_{i_2,j_2} \cdots \chi_{i_n,j_n} >_{\mathcal{S}_+},$$

Differentiating (4.32) with respect to $g_{i_1,j_1}, ..., g_{i_n,j_n}$ and evaluating the result in $\mathcal{S}_+$, we get

$$< T_1 \chi_{i_1,j_1} \chi_{i_2,j_2} \cdots \chi_{i_n,j_n} >_{\mathcal{S}_+} = \sum_{l=1}^{n} j_l < \chi_{i_1,j_1} \cdots \chi_{i_l,j_l-1} \cdots \chi_{i_n,j_n} >_{\mathcal{S}_+}$$

This is nothing but the puncture equation (2.11) already found, extended to the whole $\mathcal{S}_+$.

Had we done the same thing for (4.31) in $\mathcal{S}_-$, we would have found the analogous puncture equation for $\mathcal{T}_{-1}$ extended to $\mathcal{S}_-$.

All this confirms what we have said before about the identification of the puncture operators, primary fields and descendants. It is worth remarking that both puncture equations change their forms if we evaluate them in $\mathcal{S}$. In such a case the puncture equation is replaced by just (4.32) or, respectively, by (4.31).

Finally let us have a look at two more constraints. The next simplest W–constraints correspond the case $n = m = 1$. Eq.(4.29) gives

$$\frac{\partial}{\partial g_{1,1}} F = \sum_{m,j \geq 0} (m g_{m,j} + \delta_{m1}\delta_{j1}) \frac{\partial}{\partial g_{m,j}} F + \frac{1}{2} x^2, \tag{4.33}$$

$$\frac{\partial}{\partial g_{1,1}} F = \sum_{i,n \geq 0} (n g_{i,n} + \delta_{i1}\delta_{n1}) \frac{\partial}{\partial g_{i,n}} F + \frac{1}{2} x^2. \tag{4.34}$$

As expected, we have two dilaton equations, eq.(4.34) is compatible with the string equation (4.32), while eq.(4.33) is the dilaton equation corresponding to the string equation (4.31). So although we have two dilaton equations, we have just *one* dilaton operator $\frac{\partial}{\partial g_{1,1}}$.

5 Landau-Ginzburg representation

In the $c < 1$ models the integrable structure provides a quite effective way to compute the correlation functions and the latter admit a topological Landau-Ginzburg interpretation, [8]. Now we are going to show that this is also true in the $c = 1$ case. We will first exhibit very general formulas to calculate the correlation functions. In the full coupling space it is very difficult to solve the W–constraints exactly, so as to obtain explicit expressions for the correlators. However in some subspaces, the calculation is drastically simplified. In such cases it will be possible to explicitly see how a Landau-Ginzburg interpretation shows up.

5.1 Correlation functions on subspace $\mathcal{S}$

In this section we will consider the subspace $\mathcal{S}$, i.e. we require all the extra couplings to vanish (except $g_{1,1} = -1$). The dispersionless coupling conditions (4.25) will reduce to

$$M = \widetilde{L}, \qquad L = \hat{\sigma}(\widetilde{M}). \tag{5.1}$$

and

$$\frac{\partial}{\partial g_{i,j}} M(\zeta) + i L^{i-1} \widetilde{L}^j = \frac{\partial}{\partial g_{i,j}} \widetilde{L}(\zeta), \qquad \frac{\partial}{\partial g_{i,j}} \hat{\sigma}(\widetilde{M}(\zeta))) + j L^i \widetilde{L}^{j-1} = \frac{\partial}{\partial g_{i,j}} L(\zeta). \tag{5.2}$$

Starting from eq.(4.29), taking suitable derivatives with respect to flow parameters, then restricting to $\mathcal{S}$, we obtain

$$
\begin{aligned}
<\chi_{n,m}> &= \frac{1}{(n+1)(m+1)}\oint M^{m+1}(L)dL^{n+1}, && (5.3a)\\
<\chi_{k,l}\chi_{n,m}> &= \frac{1}{(n+1)(m+1)}\frac{\partial}{\partial g_{k,l}}\oint M^{m+1}(L)dL^{n+1} \\
&+ km(1-\delta_{l,0}) <\chi_{n+k-1,m+l-1}>, && (5.3b)\\
<\chi_{r,s}\chi_{k,l}\chi_{n,m}> &= \frac{1}{(n+1)(m+1)}\frac{\partial^2}{\partial g_{r,s}\partial g_{k,l}}\oint M^{m+1}(L)dL^{n+1} \\
&+ km(1-\delta_{l0}) <\chi_{r,s}\chi_{n+k-1,m+l-1}> \\
&+ rm(1-\delta_{s0}) <\chi_{k,l}\chi_{n+r-1,m+s-1}> && (5.3c)\\
&- rkm(m-1)(1-\delta_{l0})(1-\delta_{s0}) <\chi_{n+r+k-2,m+s+l-2}> .
\end{aligned}
$$

The most general multi–point correlation functions in $\mathcal{S}$ have been given in [4]

$$
\begin{aligned}
<\chi_{n,m}\prod_{\mu=1}^{k}\chi_{i_\mu,j_\mu}> &= \frac{1}{(n+1)(m+1)}\Big(\prod_{\mu=1}^{k}\frac{\partial}{\partial g_{i_\mu,j_\mu}}\Big)\oint M^{m+1}(L)dL^{n+1} \\
&+ \sum_{r=1}^{k}(-1)^{r+1}\frac{m!}{(m-r)!}\sum_{\rho\in S_k}\prod_{\mu=1}^{r}\Big(i_{\rho(\mu)}(1-\delta_{j_{\rho(\mu)}0})\Big) && (5.4)\\
&\cdot <\chi_{n+i_{\rho(r+1)}+\ldots+i_{\rho(k)}-r,m+j_{\rho(r+1)}+\ldots j_{\rho(k)}-r}\prod_{s=1}^{r}\chi_{i_{\rho(s)},j_{\rho(s)}}>,
\end{aligned}
$$

where ρ is an element of the symmetric group S_k. In all the above formulas the contour integral is understood around ∞.

5.2 The pure tachyonic sector in $\mathcal{S}_+$

The above equations are only formal, unless we are able to compute the explicit expressions of L and $\widetilde{L}$. To know the latter we have to solve the coupling conditions (4.25) for L and $\widetilde{L}$. This can be done explicitly for the simplest $\mathcal{M}_{p,q}$ models, [7], but the formulas are quite complicated. Since our purpose here is to unveil the LG structure of the c=1 string theory, we will limit ourselves to $\mathcal{S}_+$ and $\mathcal{S}_-$. Due to the $\mathbf{Z}_2$ symmetry of Toda hierarchy, from now on, we will only consider the case $\mathcal{S}_+$, the discussion for the small space $\mathcal{S}_-$ being exactly specular. In the parameter space $\mathcal{S}_+$, the solution to eq.(5.1) is very simple

$$
\begin{aligned}
&R = x, \qquad a_0 = t_{2,1}, \qquad a_l = 0, \qquad \forall l \geq 2\\
&b_i = x^i\sum_{r\geq i+1} r\begin{pmatrix} r-1 \\ i \end{pmatrix} t_{2,1}^{r-i-1}t_{1,r}, \qquad \forall i\geq 0, \qquad (5.5)
\end{aligned}
$$

or equivalently

$$
L = \zeta + t_{2,1} \qquad \widetilde{L} = \frac{x}{\zeta} + \sum_{r=1}^{\infty} r t_{1,r}L^{r-1}. \qquad (5.6)
$$

Plugging these expressions into eq.(5.4), we can get all the correlation functions explicitly. We are now ready to introduce a LG representation of $c = 1$ string theory. It consists of picking a potential W and representatives for the fields, and showing that they satisfy the properties of a LG topological field theory in such a way that we can identify it with $\mathcal{T}_1$. As will be apparent in a moment, the potential we have to choose is $W = \widetilde{L}$, which is non–polynomial in ζ. The representatives of the fields will be denoted $\phi_{r,s}$. They are to be identified later on with $\chi_{r,s}$, but, for the sake of clarity, we prefer to keep the two symbols distinct.

Let us, for the time being, restrict our attention to the pure tachyonic sector. We define

$$\phi_n \equiv \phi_{n,0} \equiv (L^n)'_+, \qquad \phi_{-m} \equiv \phi_{0,m} \equiv -(\widetilde{L}^m)'_-, \qquad \phi_0 = \phi_{0,0} = \frac{1}{\zeta}, \qquad n,m \geq 1. \tag{5.7}$$

Then, using eqs.(4.24a–4.24d), we can simplify the formula for three point function (5.3c), and get

$$< \phi_a\phi_b\phi_c >= -\oint_{\zeta=0} \frac{\phi_a\phi_b\phi_c}{\widetilde{L}'}. \tag{5.8}$$

where a, b, c are integers. The LHS represents the correlation functions of three tachyons. The other multi–point tachyon correlation functions can be obtained by simply taking derivatives with respect to additional couplings, for example, the four-point function is

$$< T\phi_a\phi_b\phi_c >= -\frac{\partial}{\partial t}\oint_{\zeta=0} \frac{\phi_a\phi_b\phi_c}{\widetilde{L}'}. \tag{5.9}$$

where t represents either $t_{1,n}(n \geq 1)$ or $t_{2,m}(m \geq 1)$, or x, accordingly to whether T is T_n, T_{-m} or Q.

With the above identifications, the residue formula (5.8) is the same as in the more well–known $c < 1$ Landau–Ginzburg models, except for one detail. In the standard Landau–Ginzburg theory, the integral contour surrounds all the zeroes of the superpotential, while in the present case it surrounds the origin. In $\mathcal{S}_0$ the two contour integrals coincide since the only poles of the integrand can be at zero and at ∞, but in general this equivalence has to be verified.

The residue formula (5.8) suggests that ϕ_a are the representatives of primaries of a topological LG theory. Let us find further confirmations of this suggestion. To this end let us consider the restricted integrable hierarchy. Eqs.(4.24a, 4.24b) imply that

$$\frac{\partial L(\zeta)}{\partial t_i} = 0, \qquad \frac{\partial L(\zeta)}{\partial x} = 0, \qquad \frac{\partial L(\zeta)}{\partial \widetilde{t}_1} = 1,$$

and

$$\frac{\partial \widetilde{L}(\zeta)}{\partial t_i} = \phi_i, \qquad \frac{\partial \widetilde{L}(\zeta)}{\partial x} = \phi_0, \qquad \frac{\partial \widetilde{L}(\zeta)}{\partial \widetilde{t}_1} = \phi_{-1} + \widetilde{L}', \tag{5.10}$$

where we have used the fact that $\widetilde{L} = M$ in $\mathcal{S}_+$. These equations imply that the only non-vanishing contacts between the primary fields and others are

$$\frac{\partial \phi_{-j}(\zeta)}{\partial t_{1,i}} \quad = \quad \Big[\frac{\phi_i\phi_{-j}}{\widetilde{L}'}\Big]'_-, \tag{5.11a}$$

$$
\begin{aligned}
\frac{\partial \phi_{-j}(\zeta)}{\partial x} &= \Big[\frac{\phi_0\phi_{-j}}{\widetilde{L}'}\Big]'_-, && (5.11b)\\
\frac{\partial \phi_{i}(\zeta)}{\partial t_{2,1}} &= \phi'_i, && (5.11c)\\
\frac{\partial \phi_{-j}(\zeta)}{\partial t_{2,1}} &= \phi'_{-j} + \Big[\frac{\phi_{-1}\phi_{-j}}{\widetilde{L}'}\Big]'_-. && (5.11d)
\end{aligned}
$$

In particular we see that ϕ_1 has vanishing contacts with all the other primary fields, and lowers the level of the gravitational descendants by one, i.e.

$$
\frac{\partial \phi_{-j}(\zeta)}{\partial t_{1,1}} = j\phi_{-j+1}. \tag{5.12}
$$

This confirms our identifications of the puncture operator and the primary fields.

5.3 The discrete states

Now let us turn our attention to the discrete states. We still work with $\mathcal{S}_+$. The representatives of the discrete states are defined by the Laurent series

$$
\phi_{i,j} \equiv iL^{i-1}\widetilde{L}^j - (L^i\widetilde{L}^j)'_- = (L^i\widetilde{L}^j)'_+ - jL^i\widetilde{L}^{j-1}\widetilde{L}', \qquad \phi_0 = \phi_{0,0} = \frac{1}{\zeta}. \tag{5.13}
$$

For pure tachyons this coincides with the defintion in the previous subsection. One may wonder why we choose such a peculiar combination (which is not a total derivative w.r.t. ζ, unlike the usual situation in $c<1$ case). This is uniquely determined by the requirement that the three point function have a residue formula expression. Before giving a proof, let us derive the restricted flow equations of $L, \widetilde{L}$ and M in $\mathcal{S}_+$. Using eqs.(4.7a, 4.7b) and (4.24a–4.24d), we get

$$
\begin{aligned}
\frac{\partial L(\zeta)}{\partial g_{i,j}} &= j(L^i\widetilde{L}^{j-1})_{\le 0}, && (5.14a)\\
\frac{\partial M(\zeta)}{\partial g_{i,j}} &= \delta_{j0}(L^i)' - j(L^i\widetilde{L}^j)'_- + j(L^i\widetilde{L}^{j-1})_{\le 0}M'(\zeta), && (5.14b)\\
\frac{\partial \widetilde{L}(\zeta)}{\partial g_{i,j}} &= iL^{i-1}\widetilde{L}^j - j(L^i\widetilde{L}^j)'_- + j(L^i\widetilde{L}^{j-1})_{\le 0}M'(\zeta). && (5.14c)
\end{aligned}
$$

They lead to

$$
\frac{\partial \widetilde{L}(\zeta)}{\partial g_{i,j}} = \phi_{i,j} + \frac{\partial L(\zeta)}{\partial g_{i,j}}M'(\zeta). \tag{5.15}
$$

The flow equations of the fields $\phi_{i,j}$ in $\mathcal{S}_+$ constitute a part of the so–called *contact algebra*,

$$
\begin{aligned}
\frac{\partial \phi_{k,l}(\zeta)}{\partial t_{1,i}} &= \frac{ikl}{i+k-1}\phi_{i+k-1,l-1} + \frac{i-1}{i+k-1}\Big[\frac{\phi_{i,0}\phi_{k,l}}{\widetilde{L}'}\Big]'_-,\\
\frac{\partial \phi_{k,l}(\zeta)}{\partial x} &= \Big[\frac{\phi_0\phi_{k,l}}{\widetilde{L}'}\Big]'_- - \phi_0\Big(\frac{\phi_{k,l}}{\widetilde{L}'}\Big)'_- + l\phi_0\phi_{k,l-1}, && (5.16a)\\
\frac{\partial \phi_{k,l}(\zeta)}{\partial t_{2,1}} &= \phi'_{i,j} + \Big[\frac{\phi_{0,1}\phi_{k,l}}{\widetilde{L}'}\Big]'_- + \phi_{0,1}\Big(\phi_{i,j-1} - j\frac{\phi_{k,l}}{\widetilde{L}'}\Big)'_-.
\end{aligned}
$$

In particular, we have

$$\frac{\partial \phi_{k,l}(\zeta)}{\partial t_1} = l\phi_{k,l-1}, \qquad \{k \geq 1, l \geq 0\} \oplus \{k=0, l \geq 2\}; \tag{5.17}$$
$$\frac{\partial \phi_{0,1}(\zeta)}{\partial t_1} = 0, \qquad \frac{\partial \phi_{0}(\zeta)}{\partial t_1} = 0.$$

This once again confirms that $\frac{\partial}{\partial t_{1,1}}$ is indeed a puncture operator. Furthermore, the flow equations of $L, \widetilde{L}, M$, and $\phi_{i,j}$ enable us to derive the simplified formulas for multi–point correlation functions. Then the first few multi–point functions are

$$< \phi_{n,m} > \;=\; \frac{1}{(n+1)(m+1)} \oint \widetilde{L}^{m+1} dL^{n+1}, \tag{5.18}$$

$$\begin{aligned} < \phi_{k,l}\phi_{n,m} > \;&=\; \oint (L^k \widetilde{L}^l)_- d(L^n \widetilde{L}^m) \\ &+\; \frac{km(1-\delta_{l0})}{(n+k)(m+l)} \oint \widetilde{L}^{m+l} dL^{n+k}, \end{aligned} \tag{5.19}$$

$$< \phi_{i,j}\phi_{k,l}\phi_{n,m} > \;=\; -\oint \frac{\phi_{i,j}\phi_{k,l}\phi_{n,m}}{\widetilde{L}'}. \tag{5.20}$$

In the derivation of the residue formula for the three point function, we have used eq.(5.3c), and the flow equations. However, if the correlators contains at least one primary, we can have a simpler derivation. Let us start from eq.(4.20), take one more derivative w.r.t. the coupling parameter, and make use of the equations of motion, we have

$$\begin{aligned} &< \phi_{i,j}\phi_{k,l}\phi_{n,0} >= \frac{\partial}{\partial g_{k,l}} \oint (L^i \widetilde{L}^j)_- dL^n = \oint \Big[\phi_{n,0} \frac{\partial (L^i\widetilde{L}^j)_-}{\partial g_{k,l}} - (L^i\widetilde{L}^j)'_- \frac{\partial (L^n)}{\partial g_{k,l}}\Big] d\zeta \\ &= \oint d\zeta \phi_{n,0} \Big[+jL^i\widetilde{L}^{j-1}\phi_{k,l} + l(L^k\widetilde{L}^{l-1})_{\leq 0}\Big(iL^{i-1}\widetilde{L}^j + jL^i\widetilde{L}^{j-1}M' - (L^i\widetilde{L}^j)'_-\Big)\Big] \\ &= \oint d\zeta \phi_{n,0} \Big[jL^i\widetilde{L}^{j-1}\phi_{k,l} + l(L^k\widetilde{L}^{l-1})_{\leq 0}(L^i\widetilde{L}^j)'_+\Big] \\ &= -\oint \frac{\phi_{n,0}\phi_{i,j}\phi_{k,l}}{\widetilde{L}'} + \oint d\zeta \phi_{n,0} (L^i\widetilde{L}^j)'_+ \Big[l(L^k\widetilde{L}^{l-1})_{\leq 0} + \frac{\phi_{k,l}}{\widetilde{L}'}\Big] \\ &= -\oint \frac{\phi_{n,0}\phi_{i,j}\phi_{k,l}}{\widetilde{L}'}. \end{aligned}$$

In the third step we have used the equality $M' = \widetilde{L}'$, in the last step we have used the fact that

$$\oint d\zeta \frac{f_+(\zeta)}{\widetilde{L}'} = 0, \qquad \forall f.$$

Since we have used the equality $M' = \widetilde{L}'$, in general the four–point functions are not obtainable by simply taking derivative w.r.t. the additional coupling parameter. But this is true if the fourth parameter is the coupling to tachyon or the cosmological constant, i.e.

$$< T\phi_{i,j}\phi_{k,l}\phi_{n,m} >= -\frac{\partial}{\partial t} \oint \frac{\phi_{i,j}\phi_{k,l}\phi_{n,m}}{\widetilde{L}'}. \tag{5.21}$$

where t represents either $t_{1,i}(i \geq 1)$ or $t_{2,j}(j \geq 1)$, or x, according to whether T is either T_i, T_{-j} or Q. This is further evidence that all tachyons are primary fields.

The rationale behind the construction of this subsection is as follows. Since the primary fields span the most general Laurent series of ζ, the gravitational descendants $\phi_{i,j}$ are particular combinations of the primary fields. Therefore any correlation function involving gravitational descendants can be expressed in terms of the correlation functions among only the primary fields.

5.4 Unperturbed LG

So far we have been working on $\mathcal{S}_+$. The formulas we have obtained are very suggestive of a LG framework, however they may look a bit involved especially at a first reading. For this reason, in this subsection we consider an even simpler situation, the coupling space $\mathcal{S}_0$, where the LG interpretation is particularly clear. In this case,

$$W(\zeta) = \frac{x}{\zeta}, \qquad \phi_{i,j}(\zeta) = M(i,j)x^j\zeta^{i-j-1}, \tag{5.22}$$

which shows that fields are highly degenerated, for

$$\phi_{i+n,i}(\zeta) = (n+i)x^i\phi_{n,0}, \qquad \phi_{j,j+m}(\zeta) = -(m+j)x^j\phi_{0,m}. \tag{5.23}$$

Therefore, all the correlation functions (containing discrete states) can be expressed in terms of the correlation functions among tachyons

$$< \phi_{i+n,i}\phi_\alpha\phi_\beta > \;=\; \frac{(n+i)}{ni} < \phi_{i,0}\phi_{0,i} >< \phi_{n,0}\phi_\alpha\phi_\beta >, \tag{5.24a}$$

$$< \phi_{j,m+j}\phi_\alpha\phi_\beta > \;=\; -\frac{(m+j)}{mj} < \phi_{j,0}\phi_{0,j} >< \phi_{0,m}\phi_\alpha\phi_\beta > . \tag{5.24b}$$

One can easily compute the multi–point correlation functions by means of (5.3a–5.3c) or (5.20). The results are those of section 2. From the above formulas one can easily prove once again the puncture equations and recursion relations within the LG formalism, and extract the algebra $\mathcal{R}_1$ which was introduced in section 2.1.

In this section we have shown that the extended $2d$ dispersionless Toda hierarchy subject to proper constraints and in the pure cosmological sector $\mathcal{S}_0$, admits a topological Landau-Ginzburg formulation exactly similar the $c < 1$ models. In a larger coupling space however some of the typical equations, such as the puncture equations and recursion relations, do not have in general exactly the same form as the $c < 1$ models. In such a case the correct form of these relations is embodied in the flow equations of the dispersionless Toda hierarchy and the relevant coupling conditions.

6 Conclusions

We think we can safely conclude that $\mathcal{T}_0$, $\mathcal{T}_1$ and $\mathcal{T}_{-1}$ are topological field theories both before and after perturbation by all the tachyonic operators. They have an infinite set of primaries; this seems to be an intrinsic characteristic. One may in fact ask oneself whether we can

truncate one of the above theories so as to obtain a TFT with a finite number of primaries (truncation means fixing a subset of primaries and keeping only the correlators among these primaries). The answer is however negative. One can extract from each of the above three theories infinitely many subsets containing a finite number of fields such that the metrics are invertible, but one easily realizes that associativity requires an infinite number of fields. Therefore, although one can envisage many topological subtheories of $\mathcal{T}_0$, $\mathcal{T}_1$ and $\mathcal{T}_{-1}$, they must all contain an infinite number of primaries.

There is a way to obtain submodels of the above TFT's, but it is far more sophisticated than a simple truncation and can be best understood in the framework of two–matrix model: one constrains the theory to live in a particular submanifolds of the coupling space. For example, if, after switching on the couplings $t_{1,1}, t_{1,2}, t_{1,3}, t_{2,1}, t_{2,2}$, one examines the theory along the direction $t_1 \sim x$ – the values of the remaining parameters is actually irrelevant – then one finds, [7], that the correlators of T_{2r+1} are the correlators of pure topological gravity and obey the flow equations of the KdV hierarchy. More complicated submanifolds of the coupling space generate the other KdV models and hierarchies. We quoted the KdV case because it may help us understand the nature of the two puncture operators T_0 and T_1. Since T_1 is conjugate to t_1 while T_0 is conjugate to x, in the submanifold $t_1 \sim x$ the two operators collapse to the same object, which becomes the puncture operator considered in [6].

In this complicated but significant manner the TFT studied in this paper, with its double nature, contains well–known TFT's coupled to topological gravity.

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